Hamiltonian sebuah elektron dalam quantum dot tanpa medan magnet adalah
\[
\hat{H} = \frac{\hat{p}^2}{2m^*} + \frac{1}{2} m^* \omega_0^2 \hat{r}^2 \tag{A.1}
\]
dan dengan memasukkan ke dalam persamaan Schrödinger menjadi
\[
\hat{H} \psi = E \psi \\
- \frac{\hbar^2}{2m^*} \nabla^2 + \frac{1}{2} m^* \omega_0^2 \hat{r}^2 \psi = E \psi
\]
Persamaan Schrödinger di atas dalam koordinat polar adalah
\[
- \frac{\hbar^2}{2m^*} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) \psi + \frac{1}{2} m^* \omega_0^2 r^2 \psi = E \psi \\
- \frac{\hbar^2}{2m^*} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) \psi + \left( \frac{1}{2} m^* \omega_0^2 r^2 - E \right) \psi = 0 \tag{A.2}
\]
Selanjutnya untuk mendapatkan solusi bagi persamaan (A.2), maka dilakukan pemisahan variabel
\[
\psi(r, \phi) = R(r) \phi(\phi) \tag{A.3}
\]
Disumbatitusi persamaan (A.3) ke persamaan Schrödinger memberikan
\[
\left[ \frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r} \frac{\partial R(r)}{\partial r} - \frac{m^*}{r^2} R(r) \right] + \left[ k^2 - \lambda^2 r^2 \right] R(r) + \frac{1}{\phi(\phi)} \frac{d^2 \phi(\phi)}{d \phi^2} = 0 \tag{A.4}
\]
dengan \( k = \sqrt{\frac{2m^* E}{\hbar^2}} \) berkaitan dengan swanilai \( E = \frac{\hbar^2 k^2}{2m^*} \), dan \( \lambda = \frac{m^* \omega_0}{\hbar} \) adalah besaran karakteristik osilator. Penyelesaian persamaan bervariabel sudut azimut adalah
\[
\frac{1}{\phi} \frac{d^2 \phi}{d \phi^2} = -m^2, \tag{A.5}
\]
\[
\frac{d^2 \phi}{d \phi^2} + m^2 \phi(\phi) = 0
\]
dengan \( m \) adalah bilangan kuantum azimuth (magnetic). Solusi persamaan (A.5) adalah
\[
\phi(\phi) = Ae^{im\phi} \tag{A.6}
\]
Dengan A adalah factor normalisasi. Sedangkan persamaan Schrödinger bagian radial R(r) adalah
Syarat fisisis yang harus dipenuhi oleh fungsi gelombang radial $R(r)$ adalah bahwa $r \to 0$, $R(r)$ haruslah tetap finit, dan $R(r) \to 0$ saat $r \to \infty$. Selanjutnya persamaan diferensial diferensial ini disekitar $r \approx 0$ pada orde terendah dinyatakan sebagai
\[
\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r} \frac{\partial R(r)}{\partial r} - \frac{m^2}{r^2} R(r) + (k^2 - r^2) R(r) = 0
\]
(A.7)

Penyelesaian persamaan (A.8) dilakukan dengan menggunakan Metode Frobenius yaitu
\[
R(r) = \sum_{n=0}^{\infty} a_n r^{n+s}
\]
(A.9)
\[
R'(r) = \sum_{n=0}^{\infty} (n+s) a_n r^{n+s-1}
\]
(A.10)
\[
R''(r) = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n r^{n+s-2}
\]
(A.11)

Disubtitusi persamaan (A.9), (A.10), dan (A.11) ke persamaan (A.8) diperoleh
\[
r^2 \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n r^{n+s-2} + r \sum_{n=0}^{\infty} (n+s) a_n r^{n+s-1} - m^2 \sum_{n=0}^{\infty} a_n r^{n+s} = 0
\]
(A.12)

Untuk mendapatkan solusi yang tidak trivial $a_n \neq 0$ maka suku dalam kurung yang harus sama dengan nol
\[
(n+s)(n+s-1) + (n+s) - m^2 = 0
\]
(A.13)

Untuk $n=0$ maka
\[
s(s-1) + s - m^2 = 0
\]
\[
s^2 - s + s - m^2 = 0
\]
\[
s^2 = m^2
\]
(A.14)

sehingga didapatkan
\[
R(r) \sim r^{m^2}
\]
(A.15)

Dengan alasan sama untuk asimtot dari $R(r) \to 0$ saat $r \to \infty$, maka persamaan Schrödinger bagian radial menjadi berbentuk
\[
\frac{d^2 R(r)}{dr^2} - \lambda^2 (r^2) R(r) = 0
\]
(A.16)

solusi persamaan diferensial (A.16) adalah
\[
R(r) \approx e^{-\lambda r^2/2}
\]
(A.17)
untuk mengecek kebenarannya maka dilakukan: turunan persamaan (A.17) terhadap $r$ adalah
\[
\frac{dR}{dr} = \frac{d}{dr} e^{-\frac{1}{2}r^{2}} = -\frac{\lambda}{2} (2r) e^{-\frac{1}{2}r^{2}}
\]

dan turunan kedua persamaan (A.17) terhadap r adalah

\[
\frac{d^2 R}{dr^2} = -\lambda e^{-\frac{1}{2}r^{2}} + \lambda^2 r^2 e^{-\frac{1}{2}r^{2}}
\]  

(A.18)

Subtitusi persamaan (A.19) dan (A.17) ke persamaan (A.16) akan diperoleh

\[
-\lambda e^{-\frac{1}{2}r^{2}} + \lambda^2 r^2 e^{-\frac{1}{2}r^{2}} - \lambda^2 \left(2r\right) e^{-\frac{1}{2}r^{2}} = 0
\]

\[
-\lambda e^{-\frac{1}{2}r^{2}} + \lambda^2 r^2 e^{-\frac{1}{2}r^{2}} - \lambda^2 r^2 e^{-\frac{1}{2}r^{2}} = 0
\]

(A.20)

untuk \( r \to \infty \) ruas kiri akan sama dengan nol. Jadi penyelesaian persamaan (A.16) adalah persamaan (A.17). Selanjutnya, kombinasi dua hasil di atas (Persamaan A.15 dan A.17) menghasilkan bentuk fungsi gelombang bagian radial

\[
R(r) = r^{l|l|-1} e^{-\frac{1}{2}r^{2}} F(r)
\]

(A.21)
dengan \( F(r) \) haruslah bernilai konstan tidak nol saat \( r \to 0 \) dan tidak dapat mengembang lebih cepat dari \( e^{-\frac{1}{2}r^{2}} \) saat \( r \to \infty \). Turunan persamaan (A.21) adalah

\[
\frac{dR}{dr} = \left|n\right| r^{l|l|-1} \exp\left(-\frac{\lambda r^2}{2}\right) F(r) - \lambda \left|l\right| r^{l|l|-1} \exp\left(-\frac{\lambda r^2}{2}\right) F(r) + \left|n\right|^{1} \exp\left(-\frac{\lambda r^2}{2}\right) \frac{dF(r)}{dr}
\]

\[
\frac{dR}{dr} = \left|n\right| r^{l|l|-1} \exp\left(-\frac{\lambda r^2}{2}\right) F(r) - \lambda \left|l\right| r^{l|l|-1} \exp\left(-\frac{\lambda r^2}{2}\right) F(r) +
\]

\[
r^{l|l|} \exp\left(-\frac{\lambda r^2}{2}\right) \frac{dF(r)}{dr}
\]

\[
\frac{1}{r} \frac{dR}{dr} = \left|n\right| r^{l|l|-2} \exp\left(-\frac{\lambda r^2}{2}\right) F(r) - \lambda \left|l\right| r^{l|l|-1} \exp\left(-\frac{\lambda r^2}{2}\right) F(r) +
\]

\[
r^{l|l|-1} \exp\left(-\frac{\lambda r^2}{2}\right) \frac{dF(r)}{dr}
\]

Turunan kedua persamaan (A.21) adalah
\[
\frac{d^2 R}{dr^2} = |m| |n| - 1 |l|^{n+2} \exp\left(-\frac{\lambda r^2}{2}\right) \left[ F(r) - |m| \lambda r |l|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \right]
\]

\[
+ |m| |l|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \frac{dF(r)}{dr} \left[ -\lambda (|m| + 1) |l|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \right] 
\]

\[
+ \lambda |r|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \left[ F(r) - \lambda |r|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \right] 
\]

\[
+ |m| |l|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \frac{dF(r)}{dr} 
\]

\[
+ |l|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \frac{d^2 F(r)}{dr^2} 
\]

disederhanakan menjadi

\[
\frac{d^2 R}{dr^2} = |m| |n| - 1 |l|^{n+2} \exp\left(-\frac{\lambda r^2}{2}\right) \left[ F(r) - |m| \lambda r |l|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \right]
\]

\[
+ |m| |l|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \frac{dF(r)}{dr} \left[ -\lambda (|m| + 1) |l|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \right] 
\]

\[
+ \lambda |r|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \left[ F(r) - \lambda |r|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \right] 
\]

\[
+ |m| |l|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \frac{dF(r)}{dr} 
\]

\[
+ |l|^{n+1} \exp\left(-\frac{\lambda r^2}{2}\right) \frac{d^2 F(r)}{dr^2} 
\]

Disubtitusi persamaan (A.21), (A.23) dan (A.24) ke persamaan (A.7) sehingga menjadi

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\[ |m|(|m| - 1)r^{1+1} \exp\left( -\frac{\lambda r^2}{2} \right) F(r) - |m| \lambda \ r^{1+1} \exp\left( -\frac{\lambda r^2}{2} \right) F(r) + \]
\[ |m| r^{1+1} \exp\left( -\frac{\lambda r^2}{2} \right) \frac{dF(r)}{dr} - \lambda (|m| + 1) r^{1+1} \exp\left( -\frac{\lambda r^2}{2} \right) F(r) + \]
\[ \lambda^2 r^{1+2} \exp\left( -\frac{\lambda r^2}{2} \right) F(r) - \lambda \ r^{1+1} \exp\left( -\frac{\lambda r^2}{2} \right) \frac{dF(r)}{dr} + \]
\[ |m| r^{1+1} \exp\left( -\frac{\lambda r^2}{2} \right) F(r) - \lambda r^{1+1} \exp\left( -\frac{\lambda r^2}{2} \right) \frac{dF(r)}{dr} + \]
\[ \lambda^2 r^{1+2} \exp\left( -\frac{\lambda r^2}{2} \right) F(r) - \lambda r^{1+1} \exp\left( -\frac{\lambda r^2}{2} \right) \frac{dF(r)}{dr} + \]
\[ (k^2 - r^2) r^{1+1} e^{-\lambda r^2/2} F(r) = 0 \]

Kedua ruas persamaan (A.25) dibagi dengan \( r^{1+1} \exp\left( -\frac{\lambda r^2}{2} \right) \) sehingga menjadi

\[ \frac{|m|(|m| - 1)}{r^2} F(r) - |m| \lambda F(r) + \frac{|m|}{r} \frac{dF(r)}{dr} - \lambda (|m| + 1) F(r) + \]
\[ \lambda^2 r^2 F(r) - \lambda r \frac{dF(r)}{dr} + \frac{|m|}{r} F(r) - \lambda r F(r) + \]
\[ \frac{d^2 F(r)}{dr^2} + \frac{|m|}{r^2} F(r) - \lambda F(r) + \frac{1}{r} \frac{dF(r)}{dr} + \]
\[ (k^2 - r^2) F(r) = 0 \]

Dikelompokkan kedalam suku-suku \( F(r) \), \( \frac{dF(r)}{dr} \), dan \( \frac{d^2 F(r)}{dr^2} \) sehingga menjadi

\[ \frac{d^2 F(r)}{dr^2} + \left[ \frac{2|m| + 1}{r} - 2\lambda r \right] \frac{dF(r)}{dr} - \left[ 2\lambda (|m| + 1) - k^2 \right] F(r) = 0 \]  
(A.27)

Kemudian dilakukan perubahan variable
\[ x = -\sqrt{\lambda} r \]  
(A.28)

Turunan pertama persamaan (A.28) adalah
\[ \frac{dx}{dr} = -\sqrt{\lambda} \]  
(A.29)

sehingga diperoleh
\[ \frac{d}{dr} = \frac{d}{dx} = -\sqrt{\lambda} \frac{d}{dx} \]  
(A.30)
Persamaan (A.28), (A.29), (A.30) dan (A.31) di subtitusi ke persamaan (A.27) sehingga diperoleh

\[
\lambda \frac{d^2 F(x)}{dx^2} + \left( \frac{\left|m \right| + 1}{{\lambda}} - 1 \right) \frac{dF(x)}{dx} - \frac{1}{2} \left[ \frac{\left|m \right| + 1}{{\lambda}} - \frac{k^2}{2\lambda^2} \right] F(x) = 0
\]

Kedua ruas dibagi dengan \( \lambda \) sehingga diperoleh

\[
\frac{d^2 F(x)}{dx^2} + \left[ \frac{\left|m \right| + 1}{\lambda} - 1 \right] \frac{dF(x)}{dx} - \frac{1}{2} \left[ \frac{\left|m \right| + 1}{\lambda} - \frac{k^2}{2\lambda^2} \right] F(x) = 0
\]

atau

\[
\frac{d^2 F(x)}{dx^2} + \left[ \frac{2\left|m \right| + 1}{x} - 2\left(x + \sqrt{\lambda} \right) \right] \frac{dF(x)}{dx} - \frac{1}{2} \left[ 2\left|m \right| + 1 \right] \frac{k^2}{x} \left( 2\left|m \right| + 1 \right) F(x) = 0
\]

\[
\frac{d^2 F(x)}{dx^2} + \left[ \frac{2\left|m \right| + 1 - 2x^2 - 2\sqrt{\lambda}x \left( 2\left|m \right| + 1 \right)}{x} \right] \frac{dF(x)}{dx} - \frac{1}{2} \left[ 2\left|m \right| + 2 \right] \frac{k^2}{x} \left( 2\left|m \right| + 1 \right) F(x) = 0
\]

\[
\frac{d^2 F(x)}{dx^2} - \left[ \frac{2x^2 - 1 - 2\left|m \right| + 2\sqrt{\lambda}x}{x} \right] \frac{dF(x)}{dx} - \frac{1}{2} \left[ \frac{-2k^2}{\lambda} + 4\left|m \right| + 4 \right] x + 4\sqrt{\lambda} \left( 2\left|m \right| + 4 \right) F(x) = 0
\]

Merupakan bentuk persamaan diferensial Heun Biconfluent yang memiliki bentuk umum

\[
\frac{d^2 F(x)}{dx^2} - \left( \frac{2x^2 - 1 - \alpha + \beta x}{x} \right) \frac{dF(x)}{dx} - \frac{1}{2} \left( \frac{-2\gamma + 2\alpha + 4)x + \delta + \alpha\beta + \beta}{x} \right) F(x) = 0
\]

dengan \( \alpha = 2\left|m \right|, \beta = 2\lambda, \gamma = \frac{k^2}{\lambda}, \delta = 0, x = \sqrt{\lambda}r \)
Sedangkan swanilai diperoleh dengan subtitusi nilai $\gamma - 2 - \alpha = 2n$ kedalam

\[ k^2 - \lambda \gamma = 0 \] dan subtitusi $k^2$ kedalam $E = \frac{\hbar^2 k^2}{2m^*}$ yaitu

\[ E_{nm} = \frac{\hbar^2 k^2}{2m^*} 4\lambda \left( n + \left| m \right| + 1 \right) \]

\[ = \frac{\hbar^2}{m^*} \frac{m^* \omega_o}{\hbar} \left( 2n + \left| m \right| + 1 \right) \]

\[ = \hbar \omega_o \left( 2n + \left| m \right| + 1 \right) \quad (A.34) \]
LAMPIRAN B
PENURUNAN RUMUS SWANILAI DAN SWAFUNGSI ELEKTRON DALAM QUANTUM DOT DENGAN MEDAN MAGNET LUAR

Hamiltonian sebuah elektron dalam sistem quantum dot dengan medan magnet eksternal adalah

$$\hat{H} = \frac{1}{2m} \left[ \hat{\mathbf{p}} - e \frac{\mathbf{A}(\mathbf{r})}{c} \right]^2 + \frac{1}{2} m^* \omega_0^2 \mathbf{\hat{r}}^2$$ \hspace{1cm} (B.1)

karena \( \mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r} \) maka

$$\hat{H} = \frac{1}{2m} \left[ \hat{\mathbf{p}} - e \frac{1}{c} \mathbf{B} \times \mathbf{r} \right]^2 + \frac{1}{2} m^* \omega_0^2 \mathbf{\hat{r}}^2$$

$$= \frac{1}{2m} \left[ \hat{\mathbf{p}}^2 - 2 \mathbf{p} \cdot \mathbf{B} \times \mathbf{r} + e^2 \frac{1}{4c^2} B^2 \mathbf{\hat{r}}^2 \right] + \frac{1}{2} m^* \omega_0^2 \mathbf{\hat{r}}^2$$ \hspace{1cm} (B.2)

$$= \frac{\hat{\mathbf{p}}^2}{2m^*} + \frac{e}{2mc} \hat{\mathbf{p}} \cdot (\mathbf{B} \times \mathbf{r}) + e^2 \frac{1}{8m^* c^2} B^2 \mathbf{\hat{r}}^2 + \frac{1}{2} m^* \omega_0^2 \mathbf{\hat{r}}^2$$

dengan memanfaatkan tripel produk \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \) maka

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m^*} + \frac{e}{2mc} \mathbf{B} \cdot \mathbf{r} + e^2 \frac{1}{8m^* c^2} B^2 \mathbf{\hat{r}}^2 + \frac{1}{2} m^* \omega_0^2 \mathbf{\hat{r}}^2$$

dengan \( \omega_e = \frac{eB}{mc} \)

\( \frac{1}{2} \omega_e \mathbf{\hat{L}}^2_z \) dapat dikeluarkan dari Hamiltonian karena memiliki swanilai \( \frac{1}{2} m^* \hbar \omega_e \)

sehingga Hamiltoniannya menjadi

$$H = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) + \frac{1}{2} m^* \mathbf{\hat{r}}^2 \left( \omega_0^2 + \omega_e^2 \right) + \frac{1}{2} m^* \omega_0^2 \mathbf{\hat{r}}^2$$

$$H = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) + \frac{1}{2} m^* \Omega^2 \mathbf{\hat{r}}^2 + \frac{1}{2} m^* \omega_0^2 \mathbf{\hat{r}}^2$$ \hspace{1cm} (B.3)

dengan \( \hat{\mathbf{p}} = -i\hbar \nabla, \ \nabla = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \) dan \( \Omega = \omega_0^2 + \omega_e^2 \)
Hamiltonian disubstitusi ke persamaan Schrödinger menjadi
\[ H\psi = E\psi \]
\[
- \frac{\hbar^2}{2m^*} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) \psi + \frac{1}{2} m^* \Omega^2 r^2 \psi + \frac{1}{2} m^* \omega^2 \Omega \phi^2 \psi = E\psi \quad (B.4)
\]
Kedua ruas dikalikan dengan \( -\frac{2m^*}{\hbar^2} \), sehingga diperoleh
\[
\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} - \frac{2m^*}{\hbar^2} \frac{\partial^2}{\partial \phi^2} \psi = -\frac{2m^*}{\hbar^2} \frac{\Omega^2}{\hbar^2} r^2 \psi - \frac{2m^*}{\hbar^2} \frac{1}{\hbar^2} \omega^2 \Omega \psi - \frac{2m^*}{\hbar^2} \frac{1}{\hbar^2} \omega^2 \Omega \phi^2 \psi = E \psi = 0
\]
\[
\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} - \frac{2m^*}{\hbar^2} \frac{\Omega^2}{\hbar^2} r^2 \psi - \frac{2m^*}{\hbar^2} \frac{\omega^2 \Omega}{\hbar^2} \psi - \frac{2m^*}{\hbar^2} \frac{1}{\hbar^2} \omega^2 \Omega \phi^2 \psi = E \psi = 0
\]
\[
\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} - a^2 r^2 \psi - \lambda^2 r^2 \psi + k^2 \psi = 0 \quad (B.5)
\]
dengan \( a^2 = \frac{m^*}{\hbar^2} \Omega^2 \), \( \lambda^2 = \frac{m^*}{\hbar^2} \omega^2 \Omega \), dan \( k^2 = \frac{2mE}{\hbar^2} \), kemudian kedua ruas dikalikan dengan \( r^2 \) sehingga diperoleh
\[
r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \phi^2} - a^2 r^2 \psi - \lambda^2 r^2 \psi + k^2 r^2 \psi = 0 \quad (B.6)
\]
dengan melakukan pemisahan variabel
\[
\psi(r, \phi) = R(r) \phi^\prime(\phi) \quad (B.7)
\]
disubstitusi persamaan (B.6) ke persamaan (B.5)
\[
r^2 \frac{\partial^2}{\partial r^2} R(r) + r \frac{\partial}{\partial r} \phi^\prime(\phi) R(r) + \frac{\partial^2}{\partial \phi^2} R(r) + \phi^\prime(\phi) \frac{\partial^2}{\partial \phi^2} R(r) - a^2 r^2 R(r) - \lambda^2 r^2 R(r) + k^2 r^2 R(r) = 0
\]
Kedua ruas dibagi dengan \( R(r) \phi^\prime(\phi) \) sehingga diperoleh
\[
\frac{r^2}{R(r)} \frac{\partial^2}{\partial r^2} + \frac{1}{\phi^\prime(\phi)} \frac{\partial^2}{\partial \phi^2} - a^2 r^2 - \lambda^2 r^2 + k^2 r^2 = 0 \quad (B.8)
\]
Persamaan diferensial bagian azimuth adalah
\[
\frac{1}{\phi^\prime(\phi)} \frac{d^2}{d\phi^2} = -m^2 \quad (B.9)
\]
dengan \( m \) adalah bilangan kuantum azimuth yang merupakan bilangan bulat. Solusi persamaan (B.9) adalah
\[ \phi'(\varphi) = Ae^{im\varphi} \]  
\hspace{1cm} (B.10)

Persamaan Schrödinger bagian radial adalah
\[ \frac{r^2}{R(r)} \frac{d^2 R(r)}{dr^2} + \frac{1}{R(r)} \frac{d R(r)}{dr} - m^2 - a^2 r^2 - \lambda^2 r^4 + k^2 r^2 = 0 \]  
\hspace{1cm} (B.11)

Kedua ruas persamaan (B.11) dikalikan dengan \( \frac{R'(r)}{r^2} \) sehingga diperoleh
\[ \frac{d^2 R'(r)}{dr^2} + \frac{1}{r} \frac{d R'(r)}{dr} - \frac{m^2}{r^2} R(r) - a^2 r^2 R'(r) - \lambda^2 r^4 R(r) + k^2 R'(r) = 0 \]  
\hspace{1cm} (B.12)

Asimtotik di \( r \approx 0 \) maka persamaan differensial akan menjadi
\[ \frac{d^2 R'(r)}{dr^2} + \frac{1}{r} \frac{d R'(r)}{dr} - \frac{m^2}{r^2} R(r) = 0 \]  
\hspace{1cm} (B.13)

Solusi persamaan ini adalah \( R'(r) \approx r^{\lambda m} \) (pembuktiannya dapat dilihat di lampiran A). Asimtotik di \( r \to \infty \) maka persamaan differensial akan menjadi
\[ \frac{d^2 R'(r)}{dr^2} - \left(a^2 r^2 - \lambda^2 r^4\right) R'(r) = 0 \]  
\hspace{1cm} (B.14)

Solusi persamaan ini adalah
\[ R'(r) \approx e^{-\frac{(a^2 r^2 - \lambda^2 r^4)}{2a}} \]  
\hspace{1cm} (B.15)

Turunan pertama persamaan (B.15) terhadap \( r \) adalah
\[ \frac{d R(r)}{dr} = -\frac{2a^2 r - 2\lambda^2 r}{2a} \exp\left(-\frac{(a^2 r^2 - \lambda^2 r^4)}{2a}\right) \]  
\hspace{1cm} (B.16)

Turunan kedua persamaan (B.15) terhadap \( r \) adalah
\[ \frac{d^2 R(r)}{dr^2} = -\frac{a^2 - \lambda^2}{2a} \exp\left(-\frac{(a^2 r^2 - \lambda^2 r^4)}{2a}\right) + \frac{4a^4 r^2 - 8a^2 \lambda^2 r^4 + 4\lambda^4 r^4}{4a^2} \exp\left(-\frac{(a^2 r^2 - \lambda^2 r^4)}{2a}\right) \]  
\hspace{1cm} (B.17)

Disubstitusi persamaan (B.16) dan (B.17) ke persamaan (B.14) sehingga diperoleh
\[- \frac{a^2 - \lambda^2}{a} \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) + \left( a^2 r^2 - 2\lambda^2 r^2 \frac{\lambda r^2}{a^2} \right) \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) \]
\[- (a^2 r^2 - \lambda^2 r^2) \exp \left( \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) = 0 \]

sehingga
\[
\left( a^2 r^2 - 2\lambda^2 r^2 \frac{\lambda^2 r^2}{a^2} - a^2 r^2 + \lambda^2 r^2 \frac{\lambda^2 r^2}{a^2} \right) \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) \]
\[- \frac{a^2 - \lambda^2}{a} \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) \]

Untuk \( r \to \infty \) maka ruas kiri persamaan (B.18) akan sama dengan nol, sehingga penyelesaian persamaan (B.14) adalah persamaan (B.15), kemudian kombinasi kedua hasil di atas akan diperoleh

\[ R'(r) = r|\lambda| \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) F(r) \]

Turunan pertama dari \( R'(r) \)
\[
\frac{dR'(r)}{dr} = |m| r |\lambda|^{-1} \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) F(r) \frac{2a^2 r - 2\lambda^2 r}{2a} r |\lambda| \]
\[ \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) F(r) + r |\lambda| \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) \frac{dF(r)}{dr} \]
\[
\frac{dR'(r)}{dr} = |m| r |\lambda|^{-1} \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) F(r) \frac{2a^2 r - 2\lambda^2 r}{2a} r |\lambda| \]
\[ \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) F(r) + r |\lambda| \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) \frac{dF(r)}{dr} \]
\[
\frac{dR(r)}{dr} = |m| r |\lambda|^{-1} \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) F(r) \frac{2a^2 r - 2\lambda^2 r}{2a} r |\lambda| \]
\[ + \frac{\lambda^2}{a} r |\lambda|^{-1} \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) F(r) + r |\lambda| \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) \frac{dF(r)}{dr} \]

Sehingga nilai dari \( \frac{1}{r} \frac{dR'(r)}{dr} \)
\[
\frac{1}{r} \frac{dR'(r)}{dr} = |m| r |\lambda|^{-2} \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) F(r) \frac{2a^2 r - 2\lambda^2 r}{2a} r |\lambda| \]
\[ + \frac{\lambda^2}{a} r |\lambda|^{-1} \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) F(r) + r |\lambda| \exp \left( - \frac{(a^2 r^2 - \lambda^2 r^2)}{2a} \right) \frac{dF(r)}{dr} \]

Turunan kedua \( R'(r) \)
\[
\frac{d^2 R'(r)}{dr^2} = |m|(|m| - 1)r^{l+2} \exp\left(- \left(\frac{a^2 r^2 - \lambda^2 r^2}{2a}\right)\right)F(r) - \left(\frac{2a^2 r - 2\lambda^2 r}{2a}\right)|m| r^{l+1} \\
\exp\left(- \left(\frac{a^2 r^2 - \lambda^2 r^2}{2a}\right)\right)F(r) + |m| r^{l+1} \exp\left(- \left(\frac{a^2 r^2 - \lambda^2 r^2}{2a}\right)\right)\frac{dF(r)}{dr} - a(|m| + 1)r^{l+1} \\
\exp\left(- \left(\frac{a^2 r^2 - \lambda^2 r^2}{2a}\right)\right)\frac{dF(r)}{dr} + \frac{\lambda^2}{a}(|m| + 1)r^{l+1} \exp\left(- \left(\frac{a^2 r^2 - \lambda^2 r^2}{2a}\right)\right)F(r) - \\
(2a^2 r - 2\lambda^2 r) r^{l+1} \exp\left(- \left(\frac{a^2 r^2 - \lambda^2 r^2}{2a}\right)\right)\frac{dF(r)}{dr} + \frac{\lambda^2}{a} r^{l+1} \\
(2a^2 r - 2\lambda^2 r) r^{l+1} \exp\left(- \left(\frac{a^2 r^2 - \lambda^2 r^2}{2a}\right)\right)\frac{dF(r)}{dr} + r^{l+1} \\
\exp\left(- \left(\frac{a^2 r^2 - \lambda^2 r^2}{2a}\right)\right)\frac{d^2 F(r)}{dr^2}
\]

disubtitusi persamaan (B.19),(B.21), dan (B.22) ke persamaan (B.12)
\[ |m| |m|-1 r |m|-2 \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) - \frac{(2a^2 r - 2\lambda^2 r)}{2a} |m| |m|-1 \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) + \\
+ |m| |m|-1 \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} \frac{dF(r)}{dr} - \alpha |m| |m|-1 \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) + \\
\frac{(2a^2 r - 2\lambda^2 r)}{2a} |m| |m|-1 \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) - \alpha |m| |m|-1 \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) + \\
\frac{\lambda^2}{a} (|m| |m|-1) |m| \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) - \frac{(2a^2 r - 2\lambda^2 r)}{2a} |m| |m|-1 \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) + \\
+ \frac{\lambda^2}{a} r |m| \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} \frac{dF(r)}{dr} + |m| |m|-1 \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} \frac{dF(r)}{dr} - \\
- \frac{(2a^2 r - 2\lambda^2 r)}{2a} r |m| \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} \frac{dF(r)}{dr} + r |m| \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} \frac{d^2 F(r)}{dr^2} + \\
|m| |m|-2 \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) - \alpha |m| |m|-1 \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) + \\
+ \frac{\lambda^2}{a} r |m| \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) + r |m| \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} \frac{dF(r)}{dr} - \\
m^2 r^2 r |m| \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) - a^2 r r |m| \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) - \\
\lambda^2 r^2 r |m| \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) + k^2 r |m| \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) = 0
\]

(B.23)

Kedua ruas dibagi dengan \( r |m| \exp\left\{ -\frac{(a^2 r^2 - \lambda^2 r^2)}{2a}\right\} F(r) \), sehingga akan diperoleh
\[
\begin{align*}
\left|\frac{n}{r^2}\right| &= \frac{(2a^2r - 2\lambda^2r)}{2a} |n| \frac{e^{F(r)}}{dr} - \\
&= a(2r/2a - \lambda^2) rF(r) + \frac{\lambda^2}{r^2} |n| dF(r) - \frac{2a}{a} F(r) - \frac{\lambda^2}{a} |n| dF(r) - \\
&= \left(2a^2 - 2\lambda^2r\right) \frac{dF(r)}{dr} + \frac{\lambda^2}{r^2} |n| dF(r) - \frac{2a}{a} F(r) - \frac{\lambda^2}{a} |n| dF(r) - \frac{m^2}{r^2} F(r) - a^2 \frac{dF(r)}{dr} - \lambda^2 r^2 F(r) + k^2 F(r) = 0
\end{align*}
\]
dikelompokkan menjadi
\[
\begin{align*}
\left|\frac{n}{r^2}\right| &= \frac{(2a^2 - 2\lambda^2r)}{2a} |n| F(r) + \\
&= \left(2a^2 - 2\lambda^2r\right) \frac{dF(r)}{dr} + \frac{\lambda^2}{r^2} |n| dF(r) - \frac{2a}{a} F(r) - \frac{\lambda^2}{a} |n| dF(r) - \frac{m^2}{r^2} F(r) - a^2 \frac{dF(r)}{dr} - \lambda^2 r^2 F(r) + k^2 F(r) = 0
\end{align*}
\]
disubstitusi
\[
x = -\sqrt{ar} \rightarrow r = \frac{x}{-\sqrt{a}}
\]
\[
\frac{dx}{dr} = -\sqrt{a}
\]
\[
\frac{d}{dr} = \frac{d}{dx} \frac{dx}{dr} \rightarrow \frac{d}{dr} = -\sqrt{a} \frac{d}{dx}
\]
\[
\frac{d^2}{dr} = \frac{d}{dx} \frac{d}{dr} \left(-\sqrt{a} \frac{d}{dx}\right) = a \frac{d^2}{dx^2}
\]
Disubstitusi persamaan (B.26), (B.27), (B.28), (B.29) ke dalam persamaan (B.25) dan disederhanakan menjadi
\[
\begin{align*}
&= \left\{ \left[\frac{2|m| + 1}{x} \right] \frac{\sqrt{a}}{\lambda^2 r} + 2 \frac{\lambda^2}{a} \left( \frac{\lambda^2}{x} + \frac{a^2 x}{\sqrt{a}} \right) \right\} \left( -\sqrt{a} \right) \frac{dF(x)}{dx} \\
&+ \left\{ -2a|m| - 2a - \frac{2\lambda^2 x |n|}{ax} + \frac{\lambda^4 x^2}{a^2} - \frac{\lambda^2}{ax} + k^2 - \frac{\lambda^2}{a^2} \right\} F(x) = 0
\end{align*}
\]
disederhanakan lagi menjadi
\[
a \frac{d^2 F(x)}{dx^2} + \left\{ -\frac{(2|m|+1)\sqrt{a}}{x} - \frac{2}{\sqrt{a}} \left( \frac{\lambda^2 r + a^2 x}{\sqrt{a}} \right) \right\} \frac{dF(x)}{dx} \\
+ \left\{ -2a|m| - 2a - \frac{2\lambda^2 r|m|}{\sqrt{a}x} + \frac{\lambda^4 r^2}{a^2} - \frac{\lambda^2 r}{\sqrt{a}x} + k^2 - \frac{\lambda^2 r^2}{a} \right\} F(x) = 0
\]

(B.30)

Kedua ruas persamaan (B.30) dibagi dengan \( a \), di dapatkan

\[
\frac{d^2 F(x)}{dx^2} + \left\{ -\frac{(2|m|+1)}{x} - \frac{2}{a^2} \left( \frac{\lambda^2 r + a^3 x}{a} \right) \right\} \frac{dF(x)}{dx} \\
+ \left\{ -2|m| - 2 - \frac{2\lambda^2 r|m|}{a^2 x} + \frac{\lambda^4 r^2}{a^3} - \frac{\lambda^2 r}{a^3 x} + \frac{k^2}{a} - \frac{\lambda^2 r^2}{a} \right\} F(x) = 0
\]

Disederhanakan menjadi

\[
\frac{d^2 F(x)}{dx^2} + \left\{ -\frac{(2|m|+1)}{x} - \frac{2\lambda^2 r}{a^3} - 2x \right\} \frac{dF(x)}{dx} \\
+ \left\{ \frac{k^2 - \lambda^2 r^2}{a} + \frac{\lambda^4 r^2}{a^3} - 2|m| - 2 - \frac{\lambda^2 r}{a^3 x} - \frac{2\lambda^2 r|m|}{a^3 x} \right\} F(x) = 0
\]

\[
\frac{d^2 F(x)}{dx^2} + \left\{ \frac{2|m|+1 - \frac{2\lambda^2 r}{a^2} x - 2x^2}{a^2} \right\} \frac{dF(x)}{dx} \\
+ \left\{ \frac{k^2 - \lambda^2 r^2}{a^3} + \frac{\lambda^4 r^2}{a^3} - 2|m| - 2 - \frac{\lambda^2 r}{a^3 x} - \frac{2\lambda^2 r|m|}{a^3 x} \right\} F(x) = 0
\]

dan akhirnya
$$\frac{d^2 F(x)}{dx^2} - \left[ \frac{2x^2 - 2|m| - 1 + \frac{2\lambda^2 r}{a^2} x}{x} \right] \frac{dF(x)}{dx}$$

$$+ \left[ \frac{\left( k^2 - \lambda^2 r^2 \right) a^2 + \lambda^4 r^2 - 2|m| - 2}{a^3} z - \frac{\lambda^2 r}{a^2} - \frac{2\lambda^2 r|m|}{a^2} } \right] F(x) = 0$$

dan akhirnya menjadi berbentuk persamaan differensial Heun Biconfluent (Biconfluent Heun (BCH) differential equation) yang persamaan umumnya seperti pada lampiran A

$$\frac{d^2 F(x)}{dx^2} - \left[ \frac{2x^2 - 2|m| + \frac{2\lambda^2 r}{a^2} x}{x} \right] \frac{dF(x)}{dx}$$

$$- \left[ \frac{\left( -2\left( k^2 - \lambda^2 r^2 \right) a^2 + \lambda^4 r^2 + 4|m| - 4 \right) z - \frac{2\lambda^2 r}{a^2} - \frac{4\lambda^2 r|m|}{a^2} }{2z} \right] F(x) = 0$$

(B.31)

dan memiliki penyelesaian berupa fungsi Heun Biconfluent yaitu

$$F(x) = HeunB(\alpha, \beta, \gamma, \delta, z)$$

(B.32)

dengan $$a = 2m, \beta = 2\lambda r \left( \frac{\hbar}{m^* \Omega} \right)^3, \gamma = (k^2 - \lambda^2) \left( \frac{\hbar}{m^* \Omega} \right)^3 + \lambda^4 r^2 \left( \frac{\hbar}{m^* \Omega} \right)^3, \delta = 0$$ dan $$x = -\frac{m^* \Omega}{\hbar} r.$$ Dengan syarat fungsi Heun Biconfluent akan menjadi polinomial jika dan hanya jika $$\gamma - 2 - \alpha = 2n$$ dengan $$n = 0, 1, 2, \ldots$$ adalah bilangan kuantum radial. Setelah kondisi ini terpenuhi, fungsi gelombang baru dapat di normalisasi. Dengan demikian penyelesaian lengkap diperoleh dengan menerapkan seperti pada lampiran A
\[ E_{nm} = \frac{\hbar^2}{2m^*} \left[ \frac{2m^* \Omega}{\hbar} (n + |m + 1|) - \lambda^4 r^2 \left( \frac{\hbar}{m^* \Omega} \right)^2 + \lambda^2 r^2 \right] - \frac{1}{2} |m| \hbar \omega_c \]

\[ = \hbar \Omega (n + |m + 1|) - \frac{m^* \omega_0^2 r^2}{2 \Omega} + \frac{1}{2} m^* \omega_0^2 r^2 - \frac{1}{2} |m| \hbar \omega_c \]  
\[ = \hbar \left[ \omega_0^2 + \frac{\omega_0^2}{4} (n + |m + 1|) - \frac{2m^* \omega_0^2 r^2}{(4 \omega_0^2 + \omega_c^2)} + \frac{1}{2} m^* \omega_0^2 r^2 - \frac{1}{2} |m| \hbar \omega_c \right] \]  

Jika medan magnet \( \vec{B} \) yang diterapkan pada system cukup besar maka akan mengakibatkan \( \omega_c \gg \omega_0 \), sehingga spectrum energi Fock-Darwin (B.33) akan menjadi

\[ E_{nm} = \hbar \left[ \omega_0^2 + \frac{1}{4} (n + |m + 1|) - \frac{2m^* \omega_0^2 r^2}{\omega_c^2 \left( \frac{4 \omega_0^2}{\omega_c^2} + 1 \right)} + \frac{1}{2} m^* \omega_0^2 r^2 - \frac{1}{2} |m| \hbar \omega_c \right] \]

\[ = \frac{1}{2} \hbar \omega_c (n + |m| + 1) + \frac{1}{2} m^* \omega_0^2 r^2 - \frac{1}{2} |m| \hbar \omega_c \]  
\[ = \frac{1}{2} \hbar \omega_c (n + 1) - \frac{1}{2} m^* \omega_0^2 r^2 \]
The "addition energy" needed to place an extra electron in a semiconductor quantum dot is analogous to the electron affinity for a real atom [1]. For a fixed number of electrons, small energy excitations can take these electrons to higher single-particle states. However, due to Coulomb interactions between the electrons, the addition energy is greater than the energy associated with these excitations. Both the addition energy spectrum and the excitation energy spectrum are discrete when the Fermi wavelength and the dot size are comparable. Until now a direct mapping of the observed addition energy, and the single-particle excitation energy, to a calculated spectrum has been hampered, probably due to sample specific inhomogeneities [2].

The three-dimensional, spherically symmetric potential around atoms gives rise to the shell structure 1s, 2s, 2p, 3s, 3p, ... The ionization energy has a large maximum for atomic numbers 2, 10, 18, ... Up to atomic number 23 these shells are filled sequentially, and Hund's rule determines whether a spin-down or a spin-up electron is added [3]. Vertical quantum dots have the shape of a disk with a diameter roughly 10 times the thickness [2,4]. The lateral potential has a cylindrical symmetry with a rather soft boundary profile, which can be approximated by a harmonic potential. The symmetry of this two-dimensional (2D) harmonic potential leads to a complete filling of shells for 2, 6, 12, ... electrons. The numbers in this sequence can be regarded as "magic numbers" for a 2D harmonic dot. The shell filling in this manner is previously predicted by self-consistent calculations of a circular dot [5]. In this Letter we report the observation of atomiclike properties in the conduction characteristics of a vertical quantum dot. We find an unusually large addition energy when the electron number coincides with a magic number. We can identify the quantum numbers of the single-particle states by studying the magnetic field dependence. At a sufficiently small magnetic field \( B < 0.4 \text{ T} \) we see that spin filling obeys Hund's rule. At higher magnetic fields \( B > 0.4 \text{ T} \) we observe the consecutive filling of states by spin-up and spin-down electrons, which arises from spin degeneracy.

The gated vertical quantum dot shown schematically in Fig. 1 is made from a double-barrier heterostructure (DBH). The use of well-defined heterostructure tunnel junctions allows us to vary the number of electrons in the dot \( N \) one by one from 0 to more than 40 by changing

![FIG. 1. (a) Coulomb oscillations in the current vs gate voltage at \( B = 0.4 \text{ T} \) observed for a \( D = 0.5 \mu\text{m} \) dot. (b) Addition energy vs electron number for two different dots with \( D = 0.5 \) and 0.44 \( \mu\text{m} \). The inset shows a schematic diagram of the device. The dot is located between the two heterostructure barriers.](image-url)
the gate voltage \(V_g\) [4]. The DBH consists of an undoped 12.0-μm GaAs/AlGaAs well and undoped AlGaAs barriers of thickness 9.0 and 7.5 μm (the thinner one is closest to the substrate). The source and drain contacts are made from n-GaAs and are lightly doped close to the DBH. The DBH is processed to form a mesa with top contact geometrical diameter \(D\) by using a combined dry and wet etch to a point just below the DBH region. A circular Schottky gate is placed on the side of the mesa close to the DBH [4]. We point out that the inclusion of In in the well reduces the bottom of the conduction band below the Fermi level of the contacts. This allows us to study linear transport through a vertical quantum dot. The current \(I\) flowing vertically through the dot is measured in response to a small dc voltage \(V\) applied between the contacts. Note that all the results are reproduced in both polarities for \(V\) since the device is in the linear transport regime. The samples are cooled in a dilution refrigerator down to 50 mK, although the electron temperature is estimated to be about 0.2 K.

Figure 1(a) shows the current at \(V = 150 \mu V\) as a function of \(V_g\) for a dot with \(D = 0.5 \mu m\). Clear Coulomb oscillations are observed for \(V_g > -1.6 \mu V\) with each period corresponding to a change of exactly one electron in the dot. From \(I-V\) characteristics (not shown) we can unambiguously assign absolute values of \(N\), i.e., \(N = 1\) between the first and second peaks, \(N = 2\) between the second and third peaks, etc. We find that when \(N\) becomes smaller than 20, the oscillation period depends strongly on \(N\). The increasing “irregularity” for small \(N\) has previously been reported for dots containing a few electrons [2,4], but in marked contrast we find that the irregularity in our dot is, in fact, systematic with respect to \(N\).

Figure 1(b) shows the addition energy as a function of \(N\) for two different devices. The spacing between the current peaks in Fig. 1(a) reflects the energy to add one more electron to a dot containing \(N\) electrons. For example, the energy to add the third electron to an \(N = 2\) dot can be derived from the spacing between the second and third peaks. For each value of \(N\) the factor \(\alpha\) to convert gate voltage to addition energy can be determined from the \(V_g\) dependence of the \(I-V\) characteristics [6]. The differential conductance \(dI/dV\) plotted in linear grey scale in the \(V-V_g\) plane reveals a series of diamond shaped regions associated with Coulomb blockade. The boundary of the \(N=0\) region of the Coulomb blockade is defined by the conditions that the electrochemical potential of the collector and emitter, respectively, align with the electrochemical potential \(\mu(N)\) of the dot when \(N\) and \(N + 1\) electrons are trapped in the dot. When the boundary is located at a vanishingly small \(N\), the \(N\) and \(N + 1\) peaks occur in the \(I-V_g\) characteristic. The boundary identifies the linear transport regime, and we can determine the addition energies directly from half-widths of the Coulomb diamonds. The \(\alpha\) value determined in this way, for example, in the \(D = 0.5 \mu m\) dot, varies from 57 to 42 meV/N for \(N = 1\) to 6, and then gradually decreases to 33 meV/N as \(N\) approaches 20. As \(N\) is decreased, the addition energy generally becomes larger due to the increase of the Coulomb interaction as the effective dot size is decreased. We find that the addition energy is unusually large for \(N = 2, 6, 12\) for these two devices. In eight devices with \(D\) between 0.4 and 0.54 μm the addition energy is unusually large for \(N = 2\) and 6. An unusually large addition energy for \(N = 12\) is observed in three devices. We also observe a relatively large addition energy for \(N = 4\) in most of the devices. In the remaining part of this Letter we focus on one particular \(D = 0.5 \mu m\) device. All the main features, however, have been reproduced in other devices.

The electronic states are expected to be significantly modified by a magnetic field \(B\) applied parallel to the tunneling current. We show the \(B\)-field dependence of the position of the current oscillations in Fig. 2. It is constructed from \(I-V_g\) curves for \(B\) increasing from 0 to 3.5 T in steps of 0.05 T. We can see the evolution of the first 24 current peaks. The positions of the first three peaks depend monotonously on \(B\), whereas the other peaks oscillate back and forth a number of times. The number of “wiggles” increases with \(N\). Close inspection of the figure reveals that the current peaks generally shift in pairs with \(B\). We see this even-odd effect up to \(N = 40\). Note, for instance, that around 3.5 T the peak spacing alternates between “large” for even \(N\) and “small” for odd \(N\). Intriguingly, just before entering the regime where they evolve smoothly with \(B\), the peaks making up a pair move out of phase during the last one or two wiggles (see also Ashoori et al. in Ref. [2]).

For the simplest explanation of the magic number and the \(B\) dependence we ignore, for the moment, the Coulomb interactions. The energy spectrum in a \(B\) field

![FIG. 2. Plot of the gate voltage positions of the current oscillations vs magnetic field for a dot with \(D = 0.5 \mu m\).](image-url)
can be solved analytically for a dot with a 2D radial harmonic confining potential. The energy $E_{nl}$ of a state with a radial quantum number $n (= 0, 1, 2, \ldots)$ and angular momentum quantum number $\ell (= 0, \pm 1, \pm 2, \ldots)$ is given by

$$E_{nl} = \left(2n + |\ell| + 1\right)\hbar^2 \omega_z^2 \ell^2 + \omega_z^2 \ell^2 - \frac{1}{2} \hbar \omega_c \ell.$$  \hspace{1cm} (1)

where $\hbar \omega_c$ is the cyclotron energy. Spin is neglected so each state is twofold degenerate. At $B = 0$, $E_{nl}$ has degenerate sets of states, which are separated by $\hbar \omega_c$ from each other and are completely filled for $N = 2, 6, 12, 20$, etc. These $N$ values can be regarded as magic numbers since they signify the complete filling of a shell. The unusually large addition energies we observe for $N = 2, 6, 12$ is consistent with this picture. This shell structure should persist as long as the 2D lateral potential is radially parabolic to a fairly high degree and $\hbar \omega_c$ is comparable to, or larger than, the interaction energy.

We believe that our vertical dot structures meet these conditions. However, as $N$ is increased, the potential can be significantly deformed by the effects of screening. This could be the reason why we observe the third harmonic shell only in some of the devices.

In Fig. 3(a) we plot $E_{nl}$ vs $B$ calculated for $\hbar \omega_c = 3$ meV. A single-particle state with a positive or negative $\ell$ shifts to lower or higher energies, respectively, as $B$ is initially increased from 0 T. The B-field dependence of these states gives rise to an addition energy for even $N$ that is strongly dependent on $B$. On the other hand, the addition energy for odd $N$ is determined only by the effect of Coulomb repulsion, which is responsible for lifting spin degeneracy. This should lead to the pairing of the conductance peaks, which is evident in Fig. 2.

In Fig. 3(a) we plot the energy curve for the seventh and eighth electrons with a dashed line to illustrate that these electrons undergo transitions in their quantum numbers: $(n, \ell)$ goes from $(0, 2)$ to $(0, -1)$ at 1.3 T and then to $(0, 3)$ at 2 T. These transitions are also seen in Fig. 2, demonstrating that 3 meV is a reasonable value for the confinement potential. In a similar fashion, we can identify the quantum numbers of the other electron states. A more detailed comparison can be made from Fig. 3(b), which shows the B-field dependence of the fifth, sixth, and seventh peaks measured symmetrically from $B = -5$ to 5 T. It is clear that the fifth and sixth peaks form a pair. At 1.3 T the sixth peak has a maximum, whereas the seventh peak has a minimum. This corresponds to the crossing of the third and fourth energy curves at 1.3 T in Fig. 3(a). For $\hbar \omega_c = 3$ meV, it follows that the effective dot diameter is about 100 nm. At the B field where the states last cross the single-particle states merge to form Landau levels. The single-particle excitation energy calculated, for example, at $B = 3.5$ T, is, however, still large [between 1 and 1.5 meV; see Fig. 3(a)] and significantly contributes to the addition energy for even $N$. This leads to the alternate peak spacings observed around 3.5 T in Fig. 2.

We now focus on spin filling in the second shell near $B = 0$ T. Fig. 4(a) shows the B-field dependence of the third, fourth, fifth, and sixth current peaks. The pairing of the third and fourth peaks and the fifth and sixth peaks above 0.4 T is clearly seen. However, we intriguingly find that below 0.4 T the third and fifth peaks are paired, and the fourth and sixth peaks are paired. The evolution as a pair of the third and fifth peaks for $B < 0.4$ T is continued by the third and fourth peaks for $B > 0.4$ T. Similarly, the evolution as a pair of the fourth and sixth peaks for $B < 0.4$ T is continued by the fifth and sixth peaks for $B > 0.4$ T. This reconfiguration of the pairing can be understood in terms of Hund’s rule, which is well known in atomic physics [3]. Hund’s rule favors the filling of parallel spins up to the point where the shell is half filled, and we use this to derive the B-field dependence of the electrochemical potential $\mu(N)$ given in Fig. 4(b). In a constant interaction model we can simply add a constant energy $U$ to the energy of the single-particle states to derive $\mu(N)$. To include Hund’s rule in our addition spectrum we introduce an energy $\Delta$, which represents the energy reduction due to exchange interactions between

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**FIG. 3.** (a) Calculated single-particle energy vs magnetic field for a parabolic potential with $\hbar \omega_c = 3$ meV. Each state is twofold spin degenerate. The dashed line is discussed in the text. (b) Evolution of the fifth, sixth, and seventh current peaks with B field from -5 to 5 T observed for the $D = 0.5 \mu m$ dot. The original data consists of current vs gate voltage traces for different magnetic fields, which are off-set and rotated by 90°.
electrons with parallel spins. Specifically for \( N = 4 \), the ground state energy can be lowered if the outer two electrons have parallel spins with different angular momenta rather than antiparallel spins with the same angular momentum. \( \mu(4) \) is reduced by an amount \( \Delta \) and there is a corresponding increase in \( \mu(5) \). This exchange effect is canceled in the presence of a \( B \) field when the \((0, -1)\) states, which are degenerate at \( B = 0 \) T, are split by energy \( \Delta \). This is the simplest way to account for exchange effects in a constant interaction model, however, for small \( N \) we find a remarkable agreement between what we see in Fig. 4(a) and that predicted in Fig. 4(b) if we assume \( U = 3 \) meV and \( U = 0.7 \) meV. In Fig. 4(b) we include quantum numbers \((n, \ell)\) to identify the angular momentum transitions and pictorial diagrams to illustrate the spin configurations. In our model, the addition energy for \( N = 4 \) at \( B = 0 \) T is expected to be larger by \( 2\Delta (\approx 1.4 \) meV\) than that for \( N = 3 \) and 5, and this is indeed observed in Fig. 1(b). Our model for including Hund’s rule is very simple. More rigorous Hartree-Fock calculations, as performed in Ref. [8], are needed for a quantitative comparison. Very recently Tanaka [8] and Eto [9] have actually been able to calculate addition spectra that closely reproduce our data when \( \mu_{Q} \) is comparable to or larger than the interaction energy.

For \( B < 0.4 \) T, we also see an intriguing pairing in the height of the current peaks. The fourth and sixth peaks are higher than the third and fifth peaks. In addition, as the temperature is raised from 50 mK to 1 K at \( B = 0 \) T, the fourth and sixth peaks become small, while the third and fifth peaks gradually grow. We also note that similar behaviors have been observed for \( N = 9 \) and 16 (see also the small maxima in Fig. 1(b)), which correspond to half filling of the third and fourth shells. These observations deserve further investigation.

In conclusion, we use single-electron tunneling spectroscopy to probe electronic states of a few electron vertical quantum dot atom. At zero magnetic field the addition energy reveals a shell structure associated with a 2D harmonic potential. As a function of magnetic field, current peaks evolve in pairs, due to the antiparallel filling of spin-degenerate states. Close to zero magnetic field, however, this pairing behavior is altered to favor the filling of states with parallel spins in line with Hund’s rule.

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Quantum dots
Leo Kouwenhoven and Charles Marcus

Quantum dots are man-made "droplets" of charge that can contain anything from a single electron to a collection of several thousand. Their typical dimensions range from nanometres to a few microns, and their size, shape and interactions can be precisely controlled through the use of advanced nanofabrication technology.

The physics of quantum dots shows many parallels with the behaviour of naturally occurring quantum systems in atomic and nuclear physics. Indeed, quantum dots exemplify an important trend in condensed-matter physics in which researchers study man-made objects rather than real atoms or nuclei. As in an atom, the energy levels in a quantum dot become quantized due to the confinement of electrons. With quantum dots, however, an experimentalist can scan through the entire periodic table by simply changing a voltage.

Many of these phenomena can be studied by allowing single electrons to tunnel into and out of the dot, since this reveals the quantized energy levels of the device. Experiments at our labs at the Delft University of Technology in the Netherlands and Stanford University in California have used this technique to probe various properties of quantum dots. What is most exciting is that many of the quantum phenomena observed in real atoms and nuclei - from shell structure in atoms to quantum chaos in nuclei - can be observed in quantum dots. And rather than having to study different elements or isotopes, these effects can be investigated in a quantum dot by simply changing its size or shape.

Symmetric quantum dots as artificial atoms
Many properties of quantum dots have been studied through electron tunnelling, a quantum effect that allows electrons to pass through a classically forbidden potential barrier. But electron tunnelling to and from a quantum dot is dominated by an essentially classical effect that arises from the discrete unit of charge on an electron. If the tunnelling to the dot is weak - which happens, for example, when relatively high potential barriers separate the dot from a source and drain of electrons - the number of electrons on the dot, \( N \), is a well defined integer.

Any movement of electrons through the dot requires this number to change by one. But the Coulomb repulsion between electrons means that the energy of a dot containing \( N+1 \) electrons is greater than one containing \( N \) electrons. Extra energy is therefore needed to add an electron to the dot, and no current will flow until this energy is provided by increasing the voltage. This is known as the Coulomb blockade.

To see how this works in practice, Sefo Tanaka and colleagues at NTT in Japan and one of us (LK) and co-workers at Delft have studied what happens in a symmetric quantum-dot structure (Figure 1). The structure contains a quantum dot a few hundred nanometres in diameter that is 10 nm thick and that can hold up to 100 electrons. The dot is sandwiched between two non-conducting barrier layers, which separate it from conducting material above and below. By applying a negative voltage to a metal gate around the dot, its diameter can gradually be squeezed, reducing the number of electrons on the dot - one by one - until there are none left.

This makes it possible to record the current flow as the number of electrons on the dot, and hence its energy, is varied. The Coulomb blockade leads to a series of sharp peaks in the measured current (Figure 2a). At any given peak, the number of electrons on the dot alternates between \( N \) and \( N+1 \). Between the peaks, the current is zero and \( N \) remains constant. The distance between consecutive peaks is proportional to the so-called addition energy, \( E_{add} \), which is the difference in energy between dots with \( N+1 \) and \( N \) electrons.

The simplest model to describe a quantum dot, the so-called constant-charge-interaction model, assumes that the Coulomb interaction between the electrons is independent of \( N \) and is described by the capacitance, \( C \), of the dot. In this model, the addition energy is given by \( E_{add} = e^2 / C + \Delta E \), where \( e \) is the charge on the electron, and \( \Delta E \) is the energy difference between one quantum state and the next. Adding a single electron to the dot therefore requires a constant...
changing energy, $\epsilon/C$, plus the difference in energy between the quantum states.

Despite its simplicity, this model is remarkably accurate and allows us to describe the measurements in more detail. The first peak on the graph marks the energy at which the first electron enters the dot, the second records the entry of the second electron and so on. But the spacings between the peaks are not constant, and significantly more energy is needed to add the second, sixth, and twelfth electrons.

We can picture this in terms of two-dimensional electron orbits, since the shape of the quantum dot restricts electron motion to this plane (figure 2b). The orbit with the smallest radius corresponds to the lowest energy state. This state has an angular momentum of zero and - as with atoms - can only contain two electrons with opposite spin.

This means that the charging energy, $\epsilon/C$, is enough to increase the number of electrons on the dot from one to two. But extra energy, $\Delta E$, is needed to add a third electron, since the innermost orbit will be full and the electron must go into a higher energy state. Electrons in this orbit have a angular momentum of $\pm 1$ and two spin states, which means that this shell can contain four electrons. This shell will be full once the dot contains six electrons, and so extra energy is needed to add the seventh electron.

The third shell presents a special case if the confining potential is parabolic in the radial direction, because this introduces a radial quantum number. States in this shell can have an angular momentum of 0 and a radial quantum number of 1, or an angular momentum of $\pm 2$ and a radial quantum number of 0. Together with the spin states, this means that the third shell can contain six electrons and will be full when $N=12$.

This sequence, $N=2, 6, 12, 20$ and so on, provides the “magic numbers” of electrons in a circularly symmetric harmonic potential confined to two dimensions. The energy states for such a system were calculated in the 1920s by Charles G Darwin and independently by Vladimir Fock.

The addition energy (inset to figure 2a) also shows smaller peaks at $N=4, 9$ and 16. This substructure reflects how the interactions between electrons can influence the filling of energy states. In atomic physics these effects are formulated as Hund's rules, which state that electrons enter a shell with parallel spins until the shell is half full, and then enter with opposite spin. The sequence $N=4, 9$ and 16 corresponds to a half filling of the second, third and fourth shells, where the total spin of the electrons reaches a maximum value.

This picture is summarized in a new “periodic table” of two-dimensional elements (figure 2c). The rows are shorter than those of the familiar periodic table because the dot is defined in two dimensions rather than three. But these dots are more useful than simply providing analogues of real atoms, since their larger and more controllable size allows experiments to be carried out in regimes that cannot readily be accessed for real atoms. For example, the electron orbits in both atoms and quantum dots are altered by a magnetic field, but the effect of a 1 T magnetic field on a quantum dot is comparable with the effect of a one million tesla field on a real atom. Such high magnetic fields cannot be produced in the lab.
Indeed, studies of the electronic orbits of quantum dots in a magnetic field have provided further support for the Darwin–Fock energy spectrum. And at high magnetic fields it is even possible to study the quantum Hall effect in quantum dots. For example, Ray Ashoori of the Massachusetts Institute of Technology in the US has measured how the electron states evolve as the magnetic field is increased from zero into the quantum Hall regime. And in 1991 Paul McEuen - then working with Marc Kastner at MIT and now at the University of California at Berkeley - showed that the simple model of constant Coulomb interactions no longer applies at high magnetic fields. In this regime, the interactions and confinement should be treated on an equal footing, and a more sophisticated calculation is needed to get the physics right.

We have so far focused on electron transport in the linear regime, where the voltage across the dot is small compared with the charging energy and the separation between quantum states. But this only allows us to study the ground-state energies of the artificial atoms. By increasing the voltage further, a “transport window” can be opened across the quantum dot, allowing excited states to contribute to electron transport (figure 3a). This makes it possible to measure the energy spectrum of excited states (figure 3b). Such excitation spectra, together with the effect of a magnetic field, can be calculated numerically for up to six electrons. At magnetic fields of around 1 T, for example, experimental and numerical results indicate that interesting transitions occur between the energy states on the dot. These are related to changes in the exchange energy, which accounts for the interactions between electrons with parallel spins.

**Loss of symmetry**

It is worth reiterating that the shell structures observed for both the symmetric quantum dots described above and real atoms result from symmetry. For the dots, it is the circular symmetry of the pillar that leads to the periodic table in figure 2, while for real atoms the spherical symmetry of the nuclear potential leads to the familiar periodic table of elements. While our familiarity with atoms and shell structure make symmetrical quantum systems seem ubiquitous, in fact just the opposite is true. By far the more common situation is that systems lack spatial symmetry. This is particularly the case for nanomode quantum systems. An interesting set of physical laws has recently been uncovered for quantum systems that lack symmetry. These asymmetric quantum systems are characterized by universal statistics, which were first recognized in nuclear physics and studied theoretically by Eugene Wigner, Freeman Dyson and others in the 1950s and 60s. Such universalities are now a familiar topic in mesoscopic physics.

The basic idea is that all disordered or irregularly shaped quantum systems fall into a few broad classes that are distinguished by any symmetries that remain in the system, such as symmetry under time-reversal. The quantum systems within each class share various statistical properties that describe, for example, their energy-level structure or scattering behaviour.

A surprising aspect of the universal statistics relating to quantum systems is that they seem to be connected with chaotic dynamics observed in the equivalent classical system. For example, the famous “stadium billiard” (a two-dimensional region bounded by hard walls) is a simple shape, consisting of semicircles connected by straight edges. Nonetheless, the classical trajectory of a ball bouncing in a stadium produces random motion for almost any initial condition, as proved by Leonid Bunimovich (figure 4a). The corresponding quantum system (figure 4b) has wavefunctions and energy levels that share universal features with other classically chaotic systems. This example has since become a mainstay of research into the quantum aspects of classical chaos, a subject now known as “quantum chaos.”

It is presumably the same universal behaviour of quantum chaos that leads to the random, but reproducible, fluctuations that are observed in the conduction of micro-scale metals at low temperatures. Such “mesoscopic” systems are small enough to exhibit quantum interference, but large enough to contain a random distribution of scatterers such as impurities and dislocations. These scatterers act as a source of chaos, causing electrons to move along random paths that depend on the initial conditions, just like a ball bouncing in a pinball machine.
In 1996 the statistical properties of tunnelling transport in this type of quantum dot were measured by one of us (CND) and colleagues at Stanford, and independently by Albert Chang and co-workers at Bell Labs, New Jersey, US. As in the earlier experiments, the movement of electrons is dominated by the Coulomb blockade. The conductance therefore shows a series of peaks as the gate voltage is increased (figure 6a). Two aspects of the measured peaks are expected to follow universal statistics: the distribution in peak heights and the distribution of spacings between the peaks.

Let us first consider the distribution of peak heights. At low temperatures (~100 millikelvin), the heights of the Coulomb peaks are inversely proportional to temperature and proportional, on average, to the series conductance of the two leads. However, changes in the shape of the dot or in the number of electrons it contains lead to random fluctuations in the coupling between the dot and the leads, and these should result in a universal distribution of peak heights. Rodney Jalabert, Doug Stone and Yoram Alhassid at Yale University in the US calculated this distribution in 1992, using ideas that lead to the well-known Porter-Thomas distribution of scattering widths observed in compound nuclear scattering.

Measurements of the distribution in peak heights can be made by altering the shape of the quantum dot to collect data on large numbers of different - but similar - quantum systems. Distributions measured in this way agree well with theory (figures 6b). As long as the dot generates chaotic dynamics, the distribution is independent of the size, shape and transmission of the single-channel leads. It only changes in the presence of a magnetic field, since this breaks the symmetry under time-reversal, and here again the theory matches the experimental results (figure 6c).

What about the spacing of the Coulomb peaks? A simple universal prediction can be made for any chaotic quantum dot by simply applying the approach taken earlier for symmetric dots. In this case the charging energy needed to add electrons to the dot is simply added to the universal distribution for spacings between quantum levels, which is well known from other quantum systems. Indeed, these so-called Wigner-Dyson distributions are the most famous example of the universality of quantum chaos, and seem to describe everything from the spacing between resonances in con-pond nuclear scattering to single-particle energy level spacings of classically chaotic dynamical systems.

According to the charging-energy model, the distribution of peak spacings should follow a Wigner-Dyson distribution with a width that depends on the properties of a particular dot. This theoretical distribution shows two main features: a single spike due to even-numbered electrons that only require a charging energy to enter the dot, and a broader peak due to odd-numbered electrons that require extra energy to enter the next quantum level (figure 6d).

However, this simple scheme requires a series of questionable assumptions. It supposes that the charging effects can be described by a single, slowly varying charging energy. It also assumes that the electron spins dictate how the energy levels are filled and that fluctuations in the capacitance of the dot are small. This last assumption has been called into question by Uri Sivan and colleagues at the Technion in Israel, who have investigated the peak spacings in quantum dots and wires.
Based on experimental data and numerical work with Richard Berkovits at Bar-Ilan University in Israel, Sivan and colleagues find that changes in capacitance can significantly contribute to, and in some cases dominate, the peak-spacing statistics.

Indeed, the measured distribution of the peak spacings - obtained by the Stanford group for about ten thousand peaks - disagrees with the simple theoretical model (figure 6d). In particular, the distribution shows no hint of the even-odd effects seen in the predicted distribution, challenging the assumption that each level fills first with one spin and then the other before the next level is occupied. Further work is needed to understand how the spin-Coulomb interactions and chaotic wavefunctions combine to produce this measured distribution.

Several theorists, in particular Michael Stoga, while at Riken in Japan, Berkovits, and Ned Wiegman and Kenji Hiroshi at the NEC Research Center in Princeton, US, have recently approached the problem numerically, allowing the possibility of more complicated arrangements of occupations, and that the total spin may minimize the energy of the dot. But it remains to be seen whether there are some universal statistics for these interacting systems. Recent experiments by Dan Ralph and Michael Tinkham at Harvard University in the US also emphasize the importance of spin and interaction in the properties of quantum dots. In their case, the dots are made of aluminum, which leads to superconductivity effects at low temperatures and magnetic fields.

**Future horizons**

Much of the physics observed in quantum dots has also been revealed in other quantum systems. For example, the movement of single electrons has recently been measured in carbon nanotubes and other molecular systems, and shows similar features to those observed for quantum dots. These similarities suggest that the physics of quantum dots applies to many other systems containing confined electrons.

Meanwhile, the continual development of nanotechnology will allow a greater range of artificial quantum structures to be studied. More refined theories of quantum transport will be needed to consider not only the quantization of energy and charge, but also the interactions between confined and non-confined electrons.

Some work has already started in this direction. For example, a collaboration between MIT and the Weizmann Institute in Israel and also the group in Delft with one of us (LK) has recently used quantum dots as artificial magnetic impurities embedded between metallic leads. If the dot contains an odd number of electrons, the total spin will be non-zero and electron tunneling between the dot and the leads changes the spin on the dot. The coupling between the dot and leads therefore has the effect of a spin-exchange coupling. Experiments in this regime have shown new transport features that resemble the Kondo effect in metals containing magnetic impurities.

While the Kondo effect owes its existence to interactions between many electrons, interactions can actually destroy other quantum transport phenomena. For instance, in the case of more complex circuits, the question arises whether electrons can retain their quantum mechanical properties, or whether interactions with the outside environment will lead to phase decoherence. In the last few years Marty Heiblum's group at the Weizmann Institute has carried out an interesting series of experiments aimed at measuring these coherence properties, and has observed, for instance, how placing a micro-detector near one arm of the interferometer causes decoherence. More work along these lines needs to be done before we know if it is possible to use quantum dots as the building blocks for quantum circuits.

This is a question with practical importance, since it has been suggested that such quantum circuits could form the basis of a quantum computer. However, it will take many more fundamentally interesting experiments before we get to such practical applications.

**Further reading**

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